Date 14/05/2020

D2 – Mathematics

Topic – Laplace Transform

Paper – IV

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Theorem :– If $L\{f(t)\} = F(p)$ then $L\left\{\frac{f(t)}{t}\right\} = \int_p^\infty F(x)\,dx$,

provided $\underset{t\to 0}{Lt}\, L\left\{\frac{f(t)}{t}\right\}$ exists.

<u>Proof</u>: Let $g(t) = \frac{f(t)}{t} \Rightarrow f(t) = t\,g(t)$

$$\therefore L\{f(t)\} = L\{t\,g(t)\} = -\frac{d}{dp} L\{g(t)\}$$

$$\Rightarrow F(p) = -\frac{d}{dp} L\{g(t)\}$$

Now, integrating both sides w.r.t. p from p to $p=\infty$, we have

$$\int_p^\infty F(p)\,dp = \int_p^\infty -\frac{d}{dp} L\{g(t)\}\,dp$$

$$= -\left[L\{g(t)\}\right]_p^\infty = -\underset{p\to\infty}{Lt}\, L\{g(t)\} + L\{g(t)\}$$

$$= 0 + L\{g(t)\}$$

$$\because \underset{p\to\infty}{Lt}\, L\{g(t)\} = \underset{p\to\infty}{Lt} \int_0^\infty e^{-pt} g(t)\,dt = 0 = L\{g(t)\}$$

$$\Rightarrow L\left\{\frac{f(t)}{t}\right\} = \int_p^\infty F(p)\,dp = \int_p^\infty F(x)\,dx.$$

proved

Ex. 1. Find $L\{t\cos at\}$

Solution :- We have,

$$L\{\cos at\} = \frac{p}{p^2+a^2}, \quad p>0$$

$$\therefore L\{t\cos at\} = -\frac{d}{dp}F(p)$$

$$= -\frac{d}{dp}L\{\cos at\}$$

$$= -\frac{d}{dp}\left[\frac{p}{p^2+a^2}\right]$$

$$= -\left[\frac{(p^2+a^2)\cdot 1 - p\cdot(2p)}{(p^2+a^2)^2}\right]$$

$$= -\left[\frac{p^2+a^2-2p^2}{(p^2+a^2)^2}\right]$$

$$= -\left[\frac{-p^2+a^2}{(p^2+a^2)^2}\right] = \left[\frac{p^2-a^2}{(p^2+a^2)^2}\right]$$

$$= \frac{p^2-a^2}{(p^2+a^2)^2}.$$